PHY 711: ANALYTICAL DYNAMICS

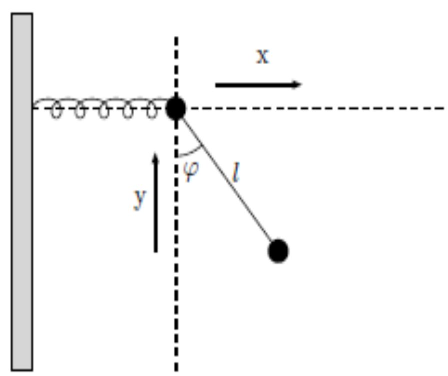
Additional Practice Problems I

Problem 1

A point mass m_1 is at the end of a horizontally placed massless spring, so that it can undergo oscillations along a horizontal line. A pendulum of length l is now suspended from the mass m_1, as shown in figure. The bob of the pendulum has mass m_2.

a) Obtain the Lagrangian and the equations of motion for the system. b) Simplify the equations of motion for small amplitudes.

c) Obtain the normal modes and eigenfrequencies of the oscillations.



Solution

This is essentially worked out in my lecture notes. For the mass m_1 at the end of the spring, the coordinates are X, Y. Y does not change since the motion of m_1 is in the horizontal plane. Thus the kinetic energy for m_1 is $\frac{1}{2}m_1\dot{X}^2$. The potential energy is $\frac{1}{2}k(X - X_0)^2$ where X_0 is the equilibrium length of the spring.

For the mass m_2 at the end of the pendulum, the coordinates are

$$x = X + l\sin\varphi, \qquad y = Y - l\cos\varphi$$

The kinetic energy is thus

$$\begin{aligned} T &= \tfrac{1}{2}m_2(\dot{x}^2 + \dot{y}^2) \\ &= \tfrac{1}{2}m_2\left(\dot{X}^2 + l^2\dot{\varphi}^2 + 2l\,\dot{\varphi}\dot{X}\cos\varphi\right) \end{aligned}$$

The Lagrangian is thus

$$L = \tfrac{1}{2}m_2\left(\dot{X}^2 + l^2\dot{\varphi}^2 + 2l\dot{\varphi}\dot{X}\cos\varphi\right) + mgl\cos\varphi + \tfrac{1}{2}m_1\dot{X}^2 - \tfrac{1}{2}k(X - X_0)^2 + \text{constant}$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{X}^2 + \tfrac{1}{2}m_2\left[l^2\dot{\varphi}^2 + 2l\dot{\varphi}\dot{X}\cos\varphi\right] + mgl\cos\varphi - \tfrac{1}{2}k(X-X_0)^2 + \text{constant}$$

This leads to the coupled equations of motion

$$(m_1+m_2)\ddot{X} + m_2 l\,\ddot{\varphi}\cos\varphi - m_2 l\,\dot{\varphi}^2\sin\varphi + k(X-X_0) = 0$$
$$\ddot{\varphi} + \frac{\ddot{X}}{l}\cos\varphi + \frac{g}{l}\sin\varphi = 0$$

For small oscillations, we take φ and $X - X_0$ to be small, and simplify these equations to first order in these quantities. (If you do this expansion in the Lagrangian, you must keep up to second order terms to get the equations correctly, since the power of the variables is reduced by one in going from the Lagrangian to the equations of motion.)

Writing $M = m_1 + m_2$, $X - X_0 = z_1$, $l\varphi = z_2$, the expansions are

$$\cos\varphi \approx 1, \qquad \sin\varphi \approx \varphi$$

so the equations of motion simplify as

$$\begin{bmatrix} M & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} + \begin{bmatrix} k & 0 \\ 0 & m_2 g/l \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

By taking the ansatz $z_1 = u\,e^{i\omega t}$, $z_2 = ve^{i\omega t}$, we find

$$\begin{bmatrix} M(\omega_1^2 - \omega^2) & -m_2\omega^2 \\ -m_2\omega^2 & m_2(\omega_2^2 - \omega^2) \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{1}$$

where we have defined $\omega_1^2 = (k/M)$, $\omega_2^2 = (g/l)$. For a nonzero solution for z_1, z_2, we need the determinant of the square matrix to vanish. Thus the allowed values of ω^2 are given by

$$\begin{vmatrix} M(\omega_1^2 - \omega^2) & -m_2\omega^2 \\ -m_2\omega^2 & m_2(\omega_2^2 - \omega^2) \end{vmatrix} = 0$$

This reduces to

$$\omega^4 - \alpha\omega^2\,(\omega_1^2 + \omega_2^2) + \alpha\,\omega_1^2\omega_2^2 = 0, \qquad \alpha = \frac{M}{m_1}$$

The solutions are given by

$$\begin{aligned} \omega_\pm^2 &= \frac{1}{2}\alpha\,(\omega_1^2+\omega_2^2) \pm \frac{1}{2}\sqrt{\alpha^2(\omega_1^2+\omega_2^2)^2 - 4\alpha\omega_1^2\omega_2^2} \\ &= \frac{1}{2}\alpha\,(\omega_1^2+\omega_2^2) \pm \frac{1}{2}\sqrt{\alpha^2(\omega_1^2-\omega_2^2)^2 + 4(\alpha^2-\alpha)\omega_1^2\omega_2^2} \end{aligned}$$

The second way of writing shows that the quantity inside the square root sign is manifestly positive since $\alpha > 1$. The first way of writing shows that the value of the square root is less than the first term, so we can conclude that $\omega_\pm^2$ are positive and real. There will be no complex or imaginary values for ω.

We can solve (1) to obtain the normal modes. The first line of that equation is

$$M(\omega_1^2 - \omega^2)\, u - m_2\omega^2\, v = 0$$

(The second line will give the same equation once we use the fact that the determinant is zero.) The solution is then given as

$$u = m_2\omega^2\, A, \qquad v = M(\omega_1^2 - \omega^2)\, A$$

for some constant A. We can choose different constants for the different choices of ω. Thus the normal modes are thus given by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} m_2\omega_+^2 \\ M(\omega_1^2 - \omega_+^2) \end{pmatrix} \left[Ae^{i\omega_+ t} + Be^{-i\omega_+ t}\right]$$
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} m_2\omega_-^2 \\ M(\omega_1^2 - \omega_-^2) \end{pmatrix} \left[Ce^{i\omega_- t} + De^{-i\omega_- t}\right]$$

where A, B, C, D are constants.

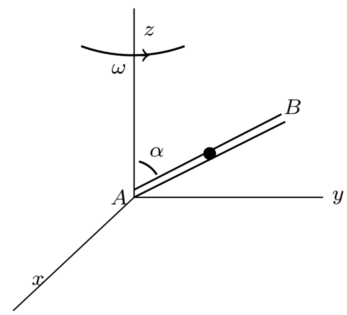
Problem 2

A thin massless open ended tube AB of length L, revolves with a constant angular velocity ω around the vertical axis CA, keeping a fixed angle α with it. A bead of mass m can slide inside the tube (neglect friction).

a) Write down the Lagrangian for this system and determine the equation of motion of the bead.

b) Determine the bead's motion if at $t = 0$ the bead is at a distance a away from point A and its initial velocity along the tube is zero.

c) What is the minimum value of the angular velocity such that the bead remains at an equilibrium position inside the tube?



<u>**Solution**</u>

Let s denote the distance along the tube from A. Then the coordinates of the bead can be written as

$$x = s\,\sin\alpha\cos\varphi, \qquad y = s\,\sin\alpha\sin\varphi, \qquad z = s\,\cos\alpha$$

Here α is fixed, and we also have $\dot{\varphi} = \omega$. The only dynamical variable is s. We find

$$\begin{aligned}
\dot{x} &= \dot{s}\,\sin\alpha\cos\varphi - s\omega\,\sin\alpha\sin\varphi \\
\dot{y} &= \dot{s}\,\sin\alpha\sin\varphi + s\omega\,\sin\alpha\cos\varphi \\
\dot{z} &= \dot{s}\,\cos\alpha
\end{aligned}$$

a) This gives the Lagrangian

$$L = \tfrac{1}{2}m(\dot{s}^2 + (\omega^2\sin^2\alpha)\,s^2) - mg\cos\alpha\,s$$

The equation of motion is

$$\ddot{s} = (\omega^2\sin^2\alpha)\,s - g\cos\alpha$$

b) Write $s = \rho + s_*$, where $s_* = (g\cos\alpha/\omega^2\sin^2\alpha)$. Then we get

$$\ddot{\rho} = (\omega^2\sin^2\alpha)\,\rho$$

with the solution $\rho = C\,e^{(\omega\sin\alpha)\,t} + De^{-(\omega\sin\alpha)\,t}$. Setting $s = a$ and $\dot{s} = 0$ at $t = 0$, we get $C = D$ and $2C = a - s_*$. Solving these, we can write s as

$$s = s_* + (a - s_*)\,\cosh\big((\omega\sin\alpha)\,t\big)$$

c) If $a = s_*$, s remains the same for all t. Thus s_* is the equilibrium position. This is an unstable equilibrium point because any deviation tends to increase with time. For the bead to remain inside the tube, we need $s \leq L$. For this to be true for the equilibrium position, we need $s_* \leq L$ or

$$\omega^2 \geq \frac{g\cos\alpha}{L\sin^2\alpha}, \qquad \text{or } \omega_{\min} = \sqrt{\frac{g\cos\alpha}{L\sin^2\alpha}}$$

Problem 3

A particle moves in a circular orbit in a force field which is radial and given by $F = -k/r^2$, where k is a positive constant, $k > 0$. If suddenly the value of k is reduced to half its original value, show that the particle's orbit becomes parabolic.

Solution

The force is essentially the gravitational force with a potential energy $V = -(k/r)$. For circular motion, we need a centripetal force equal to mv^2/r. Thus

$$\frac{mv^2}{r} = \frac{k}{r^2}$$

This gives

$$T = \tfrac{1}{2}mv^2 = \frac{k}{2r}$$

This is the initial situation where the particle has velocity $\vec{v}$ and is at a radius r. Now at the instant when we reduce k to $k/2$, the kinetic energy and r have the same values as before, but the potential energy becomes $V = -k/2r$. Thus the total energy just after the change of k is

$$E = T + V = \frac{k}{2r} - \frac{k}{2r} = 0$$

When the total energy is zero in a gravitational field, we have seen that we get a parabolic trajectory.

Problem 4

A particle moves in a central force field given by $F = kr$, where k is again positive. (This means that the force is repulsive.) Obtain the orbit of the particle, r as a function of φ, the angular variable. (*Hint*: The substitution $u = 1/r^2$ will be helpful for doing the integral.)

Solution

For the given force, the potential is $V(r) = -\frac{1}{2}kr^2$. The Lagrangian is

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \tfrac{1}{2}kr^2$$

(We have, as usual, ignored the θ-part due to conservation of angular momentum.) Writing $mr^2\dot{\varphi} = l$, we get the total energy as

$$E = \tfrac{1}{2}m\left(\dot{r}^2 + \frac{l^2}{m^2r^2}\right) - \tfrac{1}{2}kr^2$$

Solving for $\dot{r}$ and eliminating t, we get the orbit equation

$$\frac{dr}{d\varphi} = \pm \frac{r^2}{l} \sqrt{2m(E + \frac{1}{2}kr^2) - l^2/r^2}$$

Now consider $u = 1/r^2$ (not $1/r$ as we did for the Kepler problem). Differentiating and using the above equation,

$$\frac{du}{d\varphi} = -\frac{2}{r^3}\frac{dr}{d\varphi} = \mp 2\sqrt{2\frac{mE}{l^2}u - u^2 + \frac{mk}{l^2}} = \mp 2\sqrt{A^2 - \tilde{u}^2}$$

where

$$A^2 = \frac{m^2E^2}{l^4} + \frac{mk}{l^2}, \qquad \tilde{u} = u - \frac{mE}{l^2}$$

We can now do the integral using the substitution $\tilde{u} = A \sin\lambda$ to obtain $\lambda = \mp 2\varphi$ so that

$$\frac{1}{r^2} - \frac{mE}{l^2} = \mp \sqrt{\frac{m^2E^2}{l^4} + \frac{mk}{l^2}}\ \sin[2(\varphi - \varphi_0)]$$

This is the solution for the trajectories.

The interpretation of the trajectories is a little tricky. For $E > 0$, there is no problem. We get a slight modification of the straight line trajectories corresponding to zero potential energy. For $E < 0$, we have to ensure that r^2 is positive. Writing $\epsilon = -E > 0$, the formula given above becomes

$$\frac{1}{r^2} = \sqrt{\frac{m^2E^2}{l^4} + \frac{mk}{l^2}}\ [\sin[2(\varphi - \varphi_0)] - \cos\chi]$$

where

$$\cos\chi = \frac{(m\epsilon/l^2)}{\sqrt{(m\epsilon/l^2)^2 + (mk/l^2)}}$$

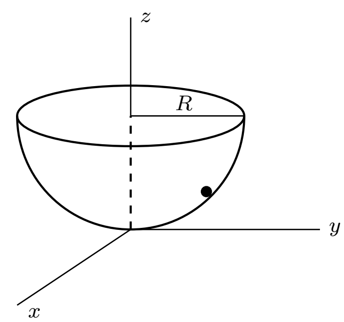
We also chose the positive sign of the square root to ensure positivity of r^2. We see that we have to restrict $\varphi - \varphi_0$ such that

$$\frac{\pi}{4} - \frac{1}{2}\chi \leq \varphi - \varphi_0 \leq \frac{\pi}{4} + \frac{1}{2}\chi$$

The orbit is open going from $r = \infty$ to $r = \infty$. Thus all orbits in this case are open.

Problem 5

A particle can slide without friction on the inner surface of a hemispherical bowl (of negligible thickness) which is resting on the ground as shown. The radius of the bowl is R. Obtain the Lagrangian and equations of motion of the particle. (You should keep in mind that the particle can have angular motion as well as radial motion. The particle cannot get off the surface, so that the vertical motion is related to the radial motion.)



Solution

Cylindrical coordinates are the simplest for this problem. Consider the horizontal plane which contains the bead; this is at height z and the circle obtained by this plane intersecting the bowl has radius r. Then we have

$$(R - z)^2 + r^2 = R^2$$

Let $R - z = \xi$. This equation reads $\xi^2 + r^2 = R^2$ and, upon differentiation, we also get $\dot{r} = -\xi\dot{\xi}/r$. We use the plane polar coordinates for the (x, y)-plane. Thus $x = r\cos\varphi$, $y = r\sin\varphi$. The kinetic term is then

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) = \tfrac{1}{2}m\left((1 + \frac{\xi^2}{r^2})\dot{\xi}^2 + r^2\dot{\varphi}^2\right) = \tfrac{1}{2}m\frac{R^2}{R^2 - \xi^2}\dot{\xi}^2 + \tfrac{1}{2}m(R^2 - \xi^2)\dot{\varphi}^2$$

The potential energy is $V = mgz = mgR - mg\xi$.

$$L = m\left[\tfrac{1}{2}\frac{R^2}{R^2 - \xi^2}\dot{\xi}^2 + \tfrac{1}{2}(R^2 - \xi^2)\dot{\varphi}^2 + g\xi\right] + \text{constant}$$

We then get

$$\begin{aligned} \frac{\partial L}{\partial \dot{\xi}} &= m\frac{R^2}{R^2 - \xi^2}\dot{\xi}, \qquad && \frac{\partial L}{\partial \dot{\varphi}} = m(R^2 - \xi^2)\,\dot{\varphi} \\ \frac{\partial L}{\partial \xi} &= m\left[\frac{R^2\xi\dot{\xi}^2}{(R^2 - \xi^2)^2} - \xi\dot{\varphi}^2 + g\right], && \frac{\partial L}{\partial \varphi} = 0 \end{aligned}$$

These give the equations of motion

$$\begin{aligned} \frac{R^2\ddot{\xi}}{R^2 - \xi^2} + \frac{R^2\,\xi\dot{\xi}^2}{(R^2 - \xi^2)^2} &= g - \xi\,\dot{\varphi}^2 \\ \frac{d}{dt}\left[(R^2 - \xi^2)\dot{\varphi}\right] &= 0 \end{aligned}$$

Problem 6

A planet is an elliptical orbit around the Sun, the orbit being described by

$$\frac{1}{r} = \frac{1 + \epsilon\,\cos\varphi}{a\,(1-\epsilon^2)}$$

where a is the semi-major axis and ϵ is the eccentricity. You may also recall that $\mu\,r^2\dot{\varphi} = l$. The planet has the maximum speed v_{max} at perihelion and minimum speed v_{min} at aphelion. Calculate the ratio v_{max}/v_{min} from the information given. Hence obtain the eccentricity of the orbit in terms of this ratio. (This is in fact one way of determining ϵ for planets. The relevant speeds can be measured, in principle, using the Doppler shift of spectral lines.)

<u>**Solution**</u>

The perihelion and aphelion correspond to the minimum and maximum value for r. From the orbit equation above, this happens when the numerator is a minimum or a maximum, i.e., at $\varphi = \pi$ and $\varphi = 0$, respectively. This gives

$$r_{max} = a(1+\epsilon), \qquad r_{min} = a(1-\epsilon)$$

Further the motion is tangential at these points, since they correspond to turning points for r. Thus the speed is $r\dot{\varphi}$ at these points. This gives

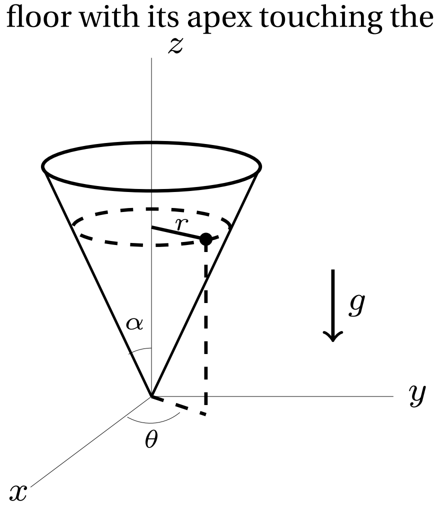
$$v_{max} = \frac{l}{\mu\,r_{min}} = \frac{l}{\mu\,a}\frac{1}{(1-\epsilon)}, \qquad v_{min} = \frac{l}{\mu\,r_{max}} = \frac{l}{\mu\,a}\frac{1}{(1+\epsilon)}$$

Thus

$$\lambda \equiv \frac{v_{max}}{v_{min}} = \frac{1+\epsilon}{1-\epsilon} \quad\Longrightarrow\quad \epsilon = \frac{\lambda-1}{\lambda+1}$$

<u>**Problem 7**</u>

A particle of mass m moves on the inner surface of a cone with opening angle α, which is placed vertically upright on the floor with its apex touching the floor, see figure.



Obtain the Lagrangian and the equations of motion for the motion of the particle. Ignore friction. (*Hint*: Cylindrical coordinates might be best. Keep in mind that there is motion corresponding to change in angle, radius and height, but not all are independent.)

Solution

We will use cylindrical coordinates, as suggested, and the (x_1, x_2) coordinates of the particle can be taken as $(r\cos\theta, r\sin\theta)$, with the relation $r = z\tan\alpha$. Thus

$$\begin{aligned} T &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \\ &= \frac{m}{2}(\dot{z}^2\sec^2\alpha \;+\; z^2\dot{\theta}^2\tan^2\alpha) \end{aligned}$$

Including the potential energy due to gravity,

$$\mathcal{L} = \frac{m}{2}(\dot{z}^2\sec^2\alpha \;+\; z^2\dot{\varphi}^2\tan^2\alpha) - mgz$$

The equations of motion are evidently

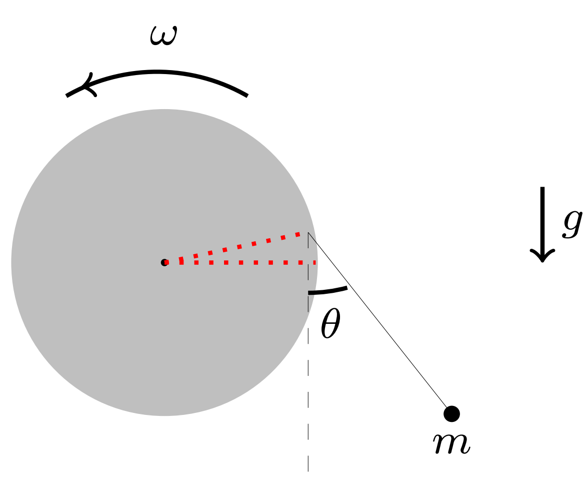
$$\sec^2\alpha\,\frac{d}{dt}\dot{z} = -g + z\dot{\theta}^2\tan^2\alpha$$

$$\frac{d}{dt}(z^2\dot{\theta}) = 0$$

Problem 8

A wheel of radius R is rotating in the vertical plane with angular velocity ω. From the rim of the wheel is suspended a pendulum of length l, with a bob of mass m. (There is a little massless axle at the point of suspension, so the string of the pendulum *will not* wind around the wheel.)

a) Obtain the Lagrangian and the equations of motion for the pendulum.



Solution

I have drawn red dotted lines from the center of the wheel to the point of suspension and along the horizontal axis. Let φ be the angle of the point of suspension relative to the x-axis; i.e., it is the angle between the red lines. Taking the center of the wheel as the origin of coordinates, we find

$$x = R\cos\varphi + l\sin\theta, \qquad y = R\sin\varphi - l\cos\theta$$

$$\dot{x} = -\omega R\sin\varphi + l\dot{\theta}\cos\theta, \qquad \dot{y} = \omega R\cos\varphi + l\dot{\theta}\sin\theta$$

The kinetic energy is found to be

$$\begin{aligned} T &= \frac{m}{2}\left[l^2\dot{\theta}^2 + \omega^2R^2 + 2\omega Rl\dot{\theta}(\sin\theta\cos\varphi - \cos\theta\sin\varphi)\right] \\ &= \frac{m}{2}\left[l^2\dot{\theta}^2 + 2\omega Rl\dot{\theta}\sin(\theta-\varphi)\right] + \text{constant} \end{aligned}$$

(Notice that the ω^2R^2-term does not involve the dynamical variable θ, so it will not contribute to the equation of motion. The potential energy is $V = mgy = -mgl\cos\theta + \text{constant}$. (What we call constant may depend on time, but is independent of the dynamical variable θ.) Thus

$$L = \frac{m}{2}\left[l^2\dot{\theta}^2 + 2\omega Rl\dot{\theta}\sin(\theta-\varphi)\right] + mgl\cos\theta$$

$$\begin{aligned} \frac{\partial L}{\partial\dot{\theta}} &= m\left[l^2\dot{\theta} + \omega Rl\sin(\theta-\varphi)\right] \\ \frac{\partial L}{\partial\theta} &= m\left[\omega Rl\dot{\theta}\cos(\theta-\varphi) - gl\sin\theta\right] \end{aligned}$$

The equation of motion is

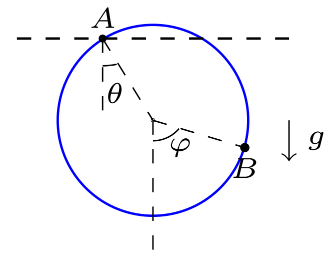
$$l^2\ddot{\theta} + \omega Rl\cos(\theta-\varphi)(\dot{\theta}-\omega) = \omega Rl\dot{\theta}\cos(\theta-\varphi) - gl\sin\theta$$

This simplifies to

$$l^2\ddot{\theta} - \omega^2Rl\ \cos(\theta-\varphi) + gl\sin\theta = 0$$

Problem 9

A hoop of uniform density and mass M and radius R (shown in blue in figure) is pivoted at a point (shown as A) on the circumference and can oscillate like a pendulum, in the vertical plane. A bead of mass m (shown as a black dot, point B) can slide frictionlessly on it.
a) Obtain the Lagrangian and the equations of motion.
b) For small oscillations of the hoop and the bead, (i.e., for small values of θ, φ in figure), find the frequencies for the normal modes of the system. (*Hint: First consider a small element of the hoop to work out its kinetic energy.*)



Problem 2

<u>**Solution**</u>

a) The motion is in a plane, so we only need x, y coordinates. We take the pivot point for the hoop as given by coordinates $(0,0)$. The coordinates for the center of the hoop are then $(R\sin\theta, -R\cos\theta)$. The position of the bead is given by

$$(x,y) = R(\sin\theta + \sin\varphi, -\cos\theta - \cos\varphi)$$

As for the hoop itself, consider a mass element $\rho d\varphi'$ where φ' is angle around the hoop. We choose an origin for this as follows. Extend the line connection point A to the center to a full diameter and measure φ' from this in the same direction as φ. The advantage is that φ' so define is independent of time as the hoop oscillates. θ can change but the position of the element relative to the specified diameter does not because we have a rigid hoop. The actual azimuthal angle for the element will be $\theta + \varphi'$. The coordinates of this element are then

$$(X,Y) = R(\sin\theta + \sin(\theta+\varphi'), -\cos\theta - \cos(\theta+\varphi'))$$

The kinetic energy for the bead is given by

$$\begin{aligned} T_{\text{bead}} &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}mR^2\left[(\cos\theta\,\dot\theta + \cos\varphi\,\dot\varphi^2 + (\sin\theta\,\dot\theta + \sin\varphi\,\dot\varphi)^2\right] \\ &= \tfrac{1}{2}mR^2\left[\dot\theta^2 + \dot\varphi^2 + 2\dot\theta\dot\varphi\,\cos(\theta-\varphi)\right] \end{aligned}$$

For the hoop, we have similarly,

$$\begin{aligned} T_{\text{hoop}} &= \tfrac{1}{2}\int d\varphi'\rho\,R^2\left[(\cos\theta\,\dot\theta + \cos(\theta+\varphi')\,\dot\theta)^2 + (\sin\theta\,\dot\theta + \sin(\theta+\varphi')\,\dot\theta)^2\right] \\ &= \tfrac{1}{2}\int d\varphi'\rho\,\dot\theta^2\,[1 + 1 + 2\cos\varphi'] = \tfrac{1}{2}(2MR^2)\dot\theta^2 \end{aligned}$$

(There is a way to obtain this result in terms of the moment of inertia of the hoop. Since we have not discussed moment of inertia in class, here I use the kinetic energy for an elemental mass and integrate. If you know about the moment of inertia, note that it is not MR^2 as it is for a hoop rotating around its geometric center. It is rotating around a point on the hoop itself, which gives an additional MR^2. If you figured out the moment of inertia correctly and used it in the formula, you do not ave to go through the integration procedure.)

The potential energy (of the form mgy) is given by

$$V = Mg(-R\cos\theta) + mg(-R\cos\theta - R\cos\varphi)$$

Combining terms, the full Lagrangian is

$$L = \tfrac{1}{2}\left[(2M+m)R^2\dot{\theta}^2 + mR^2\dot{\varphi}^2 + 2mR^2\dot{\theta}\dot{\varphi}\cos(\theta-\varphi)\right] + (M+m)gR\cos\theta + mgR\cos\varphi$$

We then find

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= (2M+m)R^2\dot{\theta} + mR^2\cos(\theta-\varphi)\,\dot{\varphi} \\
\frac{\partial L}{\partial \dot{\varphi}} &= mR^2\dot{\varphi} + mR^2\cos(\theta-\varphi)\,\dot{\theta} \\
\frac{\partial L}{\partial \theta} &= -mR^2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi) - (M+m)gR\sin\theta \\
\frac{\partial L}{\partial \varphi} &= mR^2\dot{\theta}\dot{\varphi}\sin(\theta-\varphi) - mgR\sin\varphi
\end{aligned}$$

The equations of motion become

$$\begin{aligned}
(2M+m)\ddot{\theta} + m\ddot{\varphi}\cos(\theta-\varphi) &= -m\dot{\varphi}^2\sin(\theta-\varphi) - (M+m)\frac{g}{R}\sin\theta \\
m\ddot{\varphi} + m\ddot{\theta}\cos(\theta-\varphi) &= m\dot{\theta}^2\sin(\theta-\varphi) - m\frac{g}{R}\sin\varphi
\end{aligned}$$

b) For small amplitudes, these equations can be approximated by

$$\begin{bmatrix} (2M+m) & m \\ m & m \end{bmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\varphi} \end{pmatrix} \approx \frac{g}{R}\begin{bmatrix} -(M+m) & 0 \\ 0 & -m \end{bmatrix}\begin{pmatrix} \theta \\ \varphi \end{pmatrix}$$

Taking the ansatz $\theta,\ \varphi \sim e^{i\omega t}$, we find that nontrivial solutions exist only if

$$\begin{vmatrix} (M+m)g/R - \omega^2(2M+m) & -m\omega^2 \\ -m\omega^2 & mg/R - m\omega^2 \end{vmatrix} = 0$$

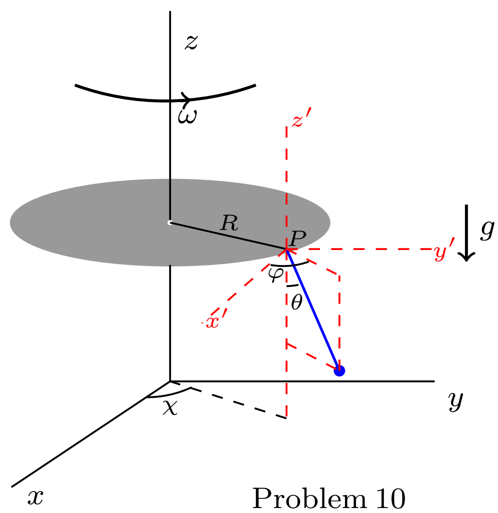
The solutions are given by

$$\omega = \pm\sqrt{\frac{M+m}{M}\frac{g}{R}}, \quad \pm\frac{1}{\sqrt{2}}\sqrt{\frac{g}{R}}$$

There will be 4 independent solutions (normal modes) corresponding to these 4 frequencies.

Problem 10

There is a disk of radius R which is rotating with angular velocity ω in the horizontal plane (say, the (x, y)-plane) around a vertical axis, see figure. A pendulum, with a string of negligible mass and a bob of mass m, is hung from a point P on the edge of the disk and can undergo oscillations, not necessarily confined to a vertical plane. Find the Lagrangian and equations of motion for the bob of the pendulum. (*Hint: Take the coordinates of the point of suspension of the pendulum on the disk as* $(X, Y, Z) = (R\cos\chi, R\sin\chi, h)$ *where* h *is fixed and the coordinates of the bob as* (x', y', z') *in terms of the red dashed coordinates and write them in terms of angles shown. Then add them vectorially to get the coordinates* (x, y, z).)



Problem 10

Solution

In terms of the (red) coordinate system with p as the origin, the coordinates of the bob of the pendulum are:

$$x' = l\sin\theta\cos\varphi, \quad y' = l\sin\theta\sin\varphi, \quad z' = -l\cos\theta$$

Adding to the vector (X, Y, Z), we find the coordinates of the bob as

$$x = R\cos\chi + l\sin\theta\cos\varphi, \quad y = R\sin\chi + l\sin\theta\sin\varphi, \quad z = h - l\cos\theta$$

Taking the time-derivatives, we find

$$\dot{x} \;\; = \;\; -R\dot{\chi}\sin\chi + l\dot{\theta}\cos\theta\cos\varphi - l\dot{\varphi}\sin\theta\sin\varphi$$

$$
\begin{aligned}
\dot{y} &= R\dot{\chi}\cos\chi + l\dot{\theta}\cos\theta\sin\varphi + l\dot{\varphi}\sin\theta\cos\varphi \\
\dot{z} &= l\dot{\theta}\sin\theta
\end{aligned}
$$

The kinetic energy is given by

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
&= \frac{1}{2}m\Big[R^2\dot{\chi}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2 + 2Rl\dot{\chi}\dot{\theta}\cos\theta\sin(\varphi - \chi) \\
&\qquad + 2Rl\dot{\chi}\dot{\varphi}\sin\theta\ \cos(\varphi - \chi)\Big] \\
&= \frac{1}{2}m\Big[l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2 + 2Rl\omega\dot{\theta}\cos\theta\sin(\varphi - \omega t) \\
&\qquad + 2Rl\omega\dot{\varphi}\sin\theta\ \cos(\varphi - \omega t)\Big] + \text{constant}
\end{aligned}
$$

where we used $\dot{\chi} = \omega$. The potential energy is $V = mgh - mgl\cos\theta$. With the Lagrangian as $L = T - V$, we find